SOLUTIONS TO EXERCISES

[Last updated 7/7/2015]

[Solutions to Problem 3, 4 need to be verified.]

Chapter 3.

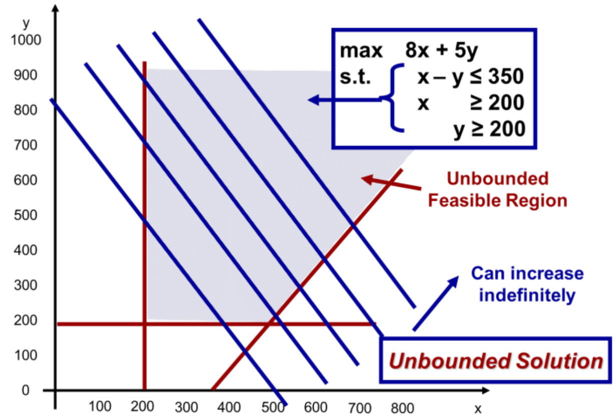
1. Three inequalities can replace two equations. A slack variable is a variable introduced to convert an inequality into an equation.
2. A system of simultaneous equations is consistent if it has at least one solution. It is redundant if one of the equations can be expressed as a linear combination of the other equations. A solution is degenerate if one or more of its components are zero. Yes; the system of equations is redundant.
3. Use slack variables to change the following inequalities into a system of equations.

$$\begin{aligned} \textit{max} \quad & 60x + 40y \\ \textit{subject to} \quad & 5x + 2y + u \le 52 \\ & x + y + v \le 20 \\ & x, y, u, v \ge 0 \end{aligned}$$

4. Change the following equations into a system of inequalities.

$$12x + 5y \le 30$$
$$x + 2y \le 12$$
$$-3x - 2y \le 0$$
$$u, v, w \ge 0$$

5. State whether the following sets of vectors are linearly dependent or independent.
   a. Linearly independent
   b. Linearly dependent
   c. Linearly dependent
   d. Linearly independent
6. The optimal basic variables are x_2 and x_3.
7. The solution space is unbounded.



8. For the following matrices, apply the pivot operation to the bolded elements.

   a. $\begin{bmatrix} -2 & 3 & \mathbf{3} \\ -2 & 4 & 2 \\ 3 & 5 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -2/3 & 1 & 1 \\ -2/3 & 2 & 0 \\ 11/3 & 4 & 0 \end{bmatrix}$

   b. $\begin{bmatrix} 1 & 2 & 2 & 8 \\ 0 & \mathbf{3} & -5 & 1 \\ 6 & 5 & 3 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 16/3 & 22/3 \\ 0 & 1 & -5/3 & 1/3 \\ 6 & 0 & 34/3 & 34/3 \end{bmatrix}$

9. The labels do not change during the computation. The labels to the left of the simplex tableau are the current basic variables and therefore change.

10. Solve the following linear programs using the Simplex Method.

    a. We introduce the slack variables s_1, s_2 to change the inequalities into a system of equations. We also rewrite the objective function as $P = 2x + y$.

$$\begin{aligned} &\textit{max} && P \\ &\textit{subject to} && 4x + y + s_1 = 150 \\ &&& 2x - 3y + s_2 = -40 \\ &&& -2x - y + P = 0 \\ &&& x, y, s_1, s_2 \geq 0 \end{aligned}$$

The initial simplex tableau is

	x	y	s_1	s_2	P	RHS
s_1	4	1	1	0	0	150
s_2	2	−3	0	1	0	−40
P	−2	−1	0	0	1	0

	x	y	s_1	s_2	P	RHS

x	1	0.25	0.25	0	0	37.5
s_2	0	-3.5	-0.5	1	0	-115
P	0	-0.5	0.5	0	1	75

	x	y	s_1	s_2	P	RHS
x	1	0	0.214	0.071	0	29.286
y	0	1	0.143	-0.286	0	32.857
P	0	0	0.571	-0.143	1	91.429

By applying the pivot operation, we obtain the max $P = 91.429$ when $x = 29.286$, $y = 32.857$, $s_1 = s_2 = 0$.

b. We introduce slack variables s_1, s_2, s_3 to change the constraints into equalities. We also rewrite the objective function as $P = 6x + 4y - 2z$.

$$\begin{aligned} &\textit{max} && P \\ &\textit{subject to} && x + y + 4z + s_1 = 20 \\ &&& -5y + 5z + s_2 = 100 \\ &&& x + 3y + z + s_3 = 400 \\ &&& -6x - 4y + 2z + P = 0 \\ &&& x, y, z, s_1, s_2, s_3 \geq 0 \end{aligned}$$

The initial Simplex tableau is

	x	y	z	s_1	s_2	s_3	P	RHS
s_1	1	1	4	1	0	0	0	20
s_2	0	-5	5	0	1	0	0	100
s_3	1	3	1	0	0	1	0	400
P	-6	-4	2	0	0	0	1	0

	x	y	z	s_1	s_2	s_3	P	RHS
x	1	1	4	1	0	0	0	20
s_2	0	-5	5	0	1	0	0	100
s_3	0	2	-3	-1	0	1	0	380

P	0	2	26	6	0	0	1	120

By applying the pivot operation, we obtain the max $P = 120$ when $y = z = s_3 = 0$, $x = 20$, $s_3 = 100$, $s_3 = 380$. Thus, the minimum value of the initial objective function is $-P = -120$.

c. We change the inequalities into a system of equations by introducing the slack variables s_1, s_2, s_3. We also rewrite the objective function as $P = 3x + 5y + 2z$.

max P

subject to $5x + y + 4z + s_1 = 50$

$x - y + z + s_2 = 150$

$2x + y + 2z + s_3 = 100$

$-3x - 5y - 2z + P = 0$

$x, y, z, s_1, s_2, s_3 \geq 0$

The initial Simplex tableau is

	x	y	z	s_1	s_2	s_3	P	RHS
s_1	5	1	4	1	0	0	0	50
s_2	1	-1	1	0	1	0	0	150
s_3	2	1	2	0	0	1	0	100
P	-3	-5	-2	0	0	0	1	0

	x	y	z	s_1	s_2	s_3	P	RHS
y	5	1	4	1	0	0	0	50
s_2	6	0	5	1	1	0	0	200
s_3	-3	0	-2	-1	0	1	0	50
P	22	0	18	5	0	0	1	250

By applying the pivot operation, we obtain the max value of the objective function $P = 250$, when $x = z = s_1 = 0$, $y = 50$, $s_2 = 200$, $s_3 = 50$.

11. Solve the following linear programming problems using the Simplex Method. Note that some variables are unrestricted.

a. We turn the objective function into a max, introduce slack variables s_1, s_2 to change the constraints into equalities, and rewrite the objective function as $P = x - y$.

$$\begin{aligned} & max \quad P \\ & subject\ to \quad x + 2y + s_1 = 100 \\ & 3x - 6y + s_2 = 650 \\ & -x + y + P = 0 \\ & y \geq 0, x \text{ unrestricted} \end{aligned}$$

Then, we convert the unrestricted variable x into two nonnegative variables x_1, x_2 such that $x = x_1 - x_2$. We have

$$\begin{aligned} & max \quad P \\ & subject\ to \quad x_1 - x_2 + 2y + s_1 = 100 \\ & 3x_1 - 3x_2 - 6y + s_2 = 650 \\ & -x_1 + x_2 + y + P = 0 \\ & x_1, x_2, y \geq 0 \end{aligned}$$

The initial Simplex tableau is

	x_1	x_2	y	s_1	s_2	P	RHS
s_1	1	-1	2	1	0	0	100
s_2	3	-3	-6	0	1	0	600
P	-1	1	1	0	0	1	0

	x_1	x_2	y	s_1	s_2	P	RHS
x_1	1	-1	2	1	0	0	100
s_2	0	0	-12	-3	1	0	300
P	0	0	3	1	0	1	100

By applying the pivot operation, we obtain the maximum value $P = 100$, when $y = s_1 = 0$, $x = x_1 - x_2 = 100$, $s_2 = 300$. Thus, the minimum value of the initial objective function is $-P = -100$.

b. We introduce slack variables s_1, s_2, s_3 in order to turn the constraints into equalities, and we rewrite the objective function $P = x + y + 2z$.

$$
\begin{aligned}
&max \quad P \\
&subject\ to \quad 5x + y + z + s_1 = 240 \\
&\qquad x - y + z + s_2 = -50 \\
&\qquad 2x + y + 2z + s_3 = 400 \\
&\qquad -x - y - 2z + P = 0 \\
&\qquad x, z, s_1, s_2, s_3 \geq 0,\ y \text{ unrestricted}
\end{aligned}
$$

We convert the unrestricted variable y into two nonnegative variables y_1, y_2 such that $y = y_1 - y_2$. We have

$$
\begin{aligned}
&max \quad P \\
&subject\ to \quad 5x + y_1 - y_2 + z + s_1 = 240 \\
&\qquad x - y_1 + y_2 + z + s_2 = -50 \\
&\qquad 2x + y_1 - y_2 + 2z + s_3 = 400 \\
&\qquad -x - y_1 + y_2 - 2z + P = 0 \\
&\qquad x, z, s_1, s_2, s_3, y_1, y_2 \geq 0
\end{aligned}
$$

The initial Simplex tableau is

	x	y_1	y_2	z	s_1	s_2	s_3	P	RHS
s_1	5	1	−1	1	1	0	0	0	240
s_2	1	−1	1	1	0	1	0	0	−50
s_3	2	1	−1	2	0	0	1	0	400
P	−1	−1	1	−2	0	0	0	1	0

	x	y_1	y_2	z	s_1	s_2	s_3	P	RHS
z	3	0	0	1	0.5	0.5	0	0	95
y_1	2	1	−1	0	0.5	−0.5	0	0	145
s_3	−6	0	0	0	−1.5	−0.5	1	0	65
P	7	0	0	0	1.5	0.5	0	1	335

By applying the pivot operation, we obtain the max of $P = 335$ when $x = s_1 = s_2 = 0$, $y = y_1 - y_2 = 145$, $z = 95$, $s_3 = 65$.

12. The objective function and constraints are

$$
min \quad 0.1x + 0.12y
$$

$$\text{subject to} \quad 6x + y \geq 250$$
$$2x + 4y \geq 350$$
$$3x + 7y \geq 420$$
$$x, y \geq 0$$

where x and y represent the number of Pill I and Pill II, respectively. We turn the objective function into a max and introduce slack variables s_1, s_2, s_3 to change the inequalities into equations. We rewrite the objective function as $P = -0.1x - 0.12y$.

$$\text{max} \quad P$$
$$\text{subject to} \quad 6x + y - s_1 = 250$$
$$2x + 4y - s_2 = 350$$
$$3x + 7y - s_3 = 420$$
$$0.1x + 0.12y + P = 0$$
$$x, y, s_1, s_2, s_3 \geq 0$$

We now rewrite the equations with $x' = -x, y' = -y$.

$$\text{max} \quad P$$
$$\text{subject to} \quad -6x' - y' - s_1 = 250$$
$$-2x' - 4y' - s_2 = 350$$
$$-3x' - 7y' - s_3 = 420$$
$$-0.1x' - 0.12y' + P = 0$$
$$x, y, s_1, s_2, s_3 \geq 0$$

The initial Simplex tableau is

	x'	y'	s_1	s_2	s_3	P	RHS
s_1	-6	-1	-1	0	0	0	250
s_2	-2	-4	0	-1	0	0	350
s_3	-3	-7	0	0	-1	0	420
P	-0.1	-0.12	0	0	0	1	0

	x'	y'	s_1	s_2	s_3	P	RHS
y'	6	1	1	0	0	0	-250

s_2	22	0	4	-1	0	0	-650
s_3	39	0	7	0	-1	0	-1330
P	0.62	0	0.12	0	0	1	-30

By applying the pivot operation, we obtain the maximum value of $P = -30$ when $x = s_1 = 0$, $y = -y' = 250$, $s_2 = 650, s_3 = 1330$. Thus, the minimum value of the initial objective function is $-P = 30$.

13. The objective function and constraints are

$$\begin{aligned} &\textit{max} \quad 7x + 5y + 3z \\ &\textit{subject to} \quad 3x + y + z \le 600 \\ &2x + 3y + 2z \le 420 \\ &2x + y + 2z \le 480 \\ &x, y, z \ge 0 \end{aligned}$$

where x, y, z represent the number of type A, B, C accessories, respectively. Then, we introduce slack variables s_1, s_2, s_3 to change the inequalities into equations and rewrite the objective function.

$$\begin{aligned} &\textit{max} \quad P \\ &\textit{subject to} \quad 3x + y + z + s_1 = 600 \\ &2x + 3y + 2z + s_2 = 420 \\ &2x + y + 2z + s_3 = 480 \\ &-7x - 5y - 3z + P = 0 \\ &x, y, z, s_1, s_2, s_3 \ge 0 \end{aligned}$$

The initial Simplex tableau is

	x	y	z	s_1	s_2	s_3	P	RHS
s_1	3	1	1	1	0	0	0	600
s_2	2	3	2	0	1	0	0	420
s_3	2	1	2	0	0	1	0	480
P	-7	-5	-3	0	0	0	1	0

	x	y	z	s_1	s_2	s_3	P	RHS
x	1	0	0.143	0.429	-0.143	0	0	197.143

y	0	1	0.571	-0.286	0.429	0	0	8.571
s_3	0	0	1.143	-0.571	-0.143	1	0	77.143
P	0	0	0.857	1.571	1.143	0	1	1422.857

By applying the pivot operation, we obtain the maximum profit $P = 1422.875$ when $z = s_1 = s_2 = 0,\ x = 197.143,\ y = 8.571,\ s_3 = 77.143$.